

Holographic Renormalization Group with Fermions and Form Fields

Jussi Kalkkinen¹ and Dario Martelli²

SISSA, Via Beirut 2 Trieste 34014, Italy
and
INFN, Sezione di Trieste

Abstract

We find the Holographic Renormalization Group equations for the holographic duals of generic gravity theories coupled to form fields and spin- $\frac{1}{2}$ fermions. Using Hamilton–Jacobi theory we discuss the structure of Ward identities, anomalies, and the recursive equations for determining the divergent terms of the generating functional. In particular, the Ward identity associated to diffeomorphism invariance contains an anomalous contribution that, however, can be solved either by a suitable counter term or by imposing a condition on the boundary fields. Consistency conditions for the existence of the dual arise, if one requires that a Callan–Symanzik type equation follows from the Hamiltonian constraint. Under mild assumptions we are able to find a class of solutions to the constraint equations. The structure of the fermionic phase space and the constraints are treated extensively for any dimension and signature.

PACS: 11.10.Kk, 11.10.Gh, 04.20.Fy, 04.62.+v

¹kalkkine@sissa.it

²dmartell@sissa.it

Contents

1	Introduction	1
2	Renormalization and Anomalies	7
3	The Holographic RG Equations	9
4	Fermions and Form Fields in Hamiltonian Theory	12
5	The Holographic Callan–Symanzik Equation	14
6	First Class Constraints as Ward Identities	16
7	Local Expansion in the Bracket Equation	19
8	Conclusions	22
A	Notation and useful formulae	23
B	Clifford algebra	26
B.1	Even dimensions	26
B.2	Odd dimensions	27
C	Fermion phase space	28

1 Introduction

In theories with diffeomorphism invariance the Poincaré generators are gauged, and the concept of a point has no invariant meaning. It becomes therefore all but impossible to define local observables. The only known way around this problem is provided by Holography [1, 2], which means (in the strong form) that a theory with diffeomorphism invariance should be describable in terms of a dual local quantum field theory defined on the boundary of space-time. In this article we investigate the conditions that a diffeomorphism invariant theory of interacting fermions and form fields has to satisfy in order that its holographic dual, in the sense defined presently, exists.

The AdS/CFT conjecture [3] has provided a very concrete example of the Holographic Principle, asserting that the physical content of a theory

of gravity in $d + 1$ dimensions should be encoded in a quantum conformal field theory living on the boundary of its anti de Sitter space-time. The correspondence has been made precise in [4, 5] where it was explained how to compute field theory observables (Green functions) in terms of AdS integrals. The central object in this correspondence is the generator of connected Green functions W

$$e^{-W[g,J]} = \int \mathcal{D}\Phi \, e^{-S[g,\Phi] - J\mathcal{O}[\Phi]} \quad (1)$$

where J is, on the QFT side, a set of external sources coupled to (composite) operators \mathcal{O} and g_{ij} is a background metric, which allows to compute the stress-energy tensor. The conjecture then states that

$$W[g, J] = S_{\text{SUGRA}}[g, J] \quad (2)$$

where the on-shell supergravity action is evaluated on an AdS solution, and it is a functional of initial values J of the fields appearing in it. The quantum field theory content is therefore encoded in a classical field theory for its sources.

This correspondence has been later generalized to non-conformal theories [6, 7, 8, 9, 10, 11, 12]. There the classical radial equations of motion on the supergravity side were interpreted as the renormalization group (RG) flow. In this way supergravity solutions provide information about RG trajectories connecting different CFT's appearing at the fixed points of the flow³. In this bulk/boundary correspondence the central object is still W , which is now a functional of the sources at a given mass scale.

In QFT W obeys a set of Ward identities, which ensure that classical symmetries are preserved at the quantum level and, consequently, constrain the counter terms that can be used in renormalization. For instance, the Ward identity of (broken) scale invariance is the Callan–Symanzik equation, which is a particular form of RG equation. Much of the information contained in this structure can be traced back to divergences. Regularization of these infinities introduces a scale, and the requirement that the renormalized quantities are independent of the cutoff produces the RG equation. In this article, we will discuss how the holographic renormalization group (HRG)

³This mechanism was discussed in [13] in the framework of Polyakov's proposal for holographic noncritical strings [14, 15].

manages, generically, to produce these features, but also how, and why, it may sometimes seem to fail if the theory includes interacting fermions.

As Holography relates theories in d and $d + 1$ dimensions it is useful to resort to Hamiltonian formalism, where the transverse “time” coordinate plays a distinguished role. As we are particularly interested in the on-shell action it is rather natural, as proposed in [16], to use Hamilton–Jacobi theory. There are also some conceptual reasons for doing so: Most importantly, on the QFT side, one eventually needs a formulation in terms of first order equations, as they can be interpreted as RG flow equations. Also, on the gravity side, it is natural to expect the Hamiltonian formulation to arise [17], as one considers classical gravity as a WKB limit of quantum gravity (String/M-Theory).

In the Hamiltonian formalism one chooses a coordinate t , with tangent vector field e_0 , to parameterize the evolution of the initial value hypersurface, the “boundary”. It is sufficient to characterize this hypersurface by giving its normal direction n with norm $n \cdot n = \pm 1$. Depending on this sign, denoted η , one is considering either time-like or space-like boundaries. This will produce sign changes with respect to usual Minkowskian Hamiltonian theory, and we will most efficiently keep track of them by leaving η unspecified. Otherwise we follow the standard Hamiltonian reduction. Writing the full metric

$$ds^2 = (\eta N^2 + N_i N^i) dt^2 + 2N_i dt dx^i + g_{ij} dx^i dx^j \quad (3)$$

the canonical (ADM) gravitational action coupled to matter has the form

$$S = \int dt \left\{ p \dot{q} - \int d^d x \, N \mathcal{H}_\perp + N^i \mathcal{H}_i + \Theta^A \mathcal{G}_A \right\} \quad (4)$$

where $p\dot{q}$ is a shorthand for the kinetic term of the dynamical degrees of freedom, N and N^i are the lapse and shift functions, which together with Θ^A are the Lagrange multipliers for the constraints⁴

$$\mathcal{H}_\perp \approx \mathcal{H}_i \approx \mathcal{G}_A \approx 0 . \quad (5)$$

These constraints generate, in the bulk theory, orthogonal deformations and diffeomorphisms on the initial value hypersurface and possible additional local symmetries, such as gauge, local Lorentz or supersymmetries.

⁴These are first class constraints. We will consider also second class constraints. They appear in the presence of fermionic fields that enter the action linearly.

In the Hamilton–Jacobi theory one makes a canonical transformation such that the new phase space coordinates are constants of motion. In gravitational theories the generating functional F of the canonical transformation must satisfy simultaneously the constraints

$$\mathcal{H}_\perp(q, \frac{\delta F}{\delta q}) = 0 \quad (6)$$

$$\mathcal{H}_i(q, \frac{\delta F}{\delta q}) = 0 \quad (7)$$

$$\mathcal{G}_A(q, \frac{\delta F}{\delta q}) = 0 . \quad (8)$$

The Hamilton principal function $F[q]$ is actually just the classical action evaluated at some given time t for fixed boundary values $q(t)$. The momenta can be calculated from

$$p = \frac{\delta F[q]}{\delta q} . \quad (9)$$

This gives us the action; if we want to calculate the full equations of motion, we have to consider also

$$\dot{q} = \{q, H\}_{\text{DB}} , \quad (10)$$

where $\{\cdot, \cdot\}_{\text{DB}}$ denote the Dirac brackets. The holographic correspondence allows us to give a dual field theory interpretation to the equations above.

Halving the degree of the equations of motion by using Hamilton–Jacobi equations for F , in spite of giving a rather appealing form for application to holography, boils down to having a high degree of arbitrariness in F . Equations (6), (7), and (8) have in fact a huge number of solutions, many of them unacceptable on physical grounds. This was discussed⁵ in [19], for a scalar field coupled to gravity. One should keep in mind also the fact that not every gravity solution corresponds to deformations of a CFT by adding an operator to the fixed point Lagrangian, as was pointed out in [20, 21]. Rather, some of them may correspond to an altogether different vacuum of the same theory, where an operator has acquired a nontrivial vacuum expectation value⁶. In general, only a subset of solutions will admit a direct

⁵There are some comments in this direction in [18], as well.

⁶For a clear exposition of this point see for instance [22]. This was illustrated explicitly in [19] in connection to the Hamilton–Jacobi formalism.

physical interpretation [23]. To ensure a physically acceptable result one specifies the asymptotic (near the AdS boundary) behaviour of the solutions. If one wishes to solve the equations for a coupled system, with a dynamical metric or interacting sources, in a perturbative fashion, the back-reaction will involve the flow equations, which determine the scaling behaviour of the fields, modifying then the general ansatz for F — even for its local part, which is the only one computable in general.

Finally, let us briefly comment on how the QFT divergences arise on the gravity side. It is well known from explicit calculations that the Einstein–Hilbert action (with the cosmological constant and the Gibbons–Hawking term) is divergent when evaluated on asymptotically AdS solutions. For instance in the coordinate system

$$ds^2 = \frac{1}{t^2}(dt^2 + g_{ij}^R dx^i dx^j) \quad (11)$$

the determinant gives $\sqrt{g} = t^{-d}\sqrt{g^R}$, and $R_{ij} = t^2 R_{ij}^R$, so that divergences arise in the limit $t \rightarrow 0$ (asymptotically AdS boundary) for terms containing up to $[\frac{d-1}{2}]$ Ricci’s. The existence of such solutions in this context is a physical requirement: It means that the RG has fixed points. In general, including additional terms in the bulk action will introduce further divergences.

In this article we shall consider the fermions and form fields with some apriorously fixed anomalous dimensions at the UV conformal point, and eventually consider how general the results are. This is tantamount to restricting the analysis to a particular subset of operators used to flow from the UV CFT. In even boundary dimension there can be additional logarithmically divergent terms that, in the Hamilton–Jacobi context, can arise only from *nonlocal* contributions. One should isolate the terms in F that are divergent in the CFT limit: These terms will give rise to the beta functions, while the rest will give the renormalized generating functional.

It turns out that – in analogy with what is achieved by QFT Ward identities – these local divergent terms, which we will call S_{div} , are fixed by the constraints in the classical theory, and can indeed be determined recursively order by order according to their degree of divergence. This recursive procedure can be expressed as “descent equations” and is computationally equivalent to the counter term generating algorithm proposed in [24, 25] in

the context of pure AdS gravity⁷. Performing this calculation one sees that (7) and (8) will produce Ward identities (diffeomorphism and gauge symmetries), while (6) provides a recursive equation for S_{div} , formally very similar to the perturbative expansion of the master equation, that in addition will give rise to a Callan–Symanzik-type equation for W .

One should naturally try to choose a renormalization procedure that preserves the symmetries. A failure to fulfill this requirement would give rise to an anomaly. Anomalies in the boundary theory may arise from bulk contributions to the Hamiltonian constraints. This may happen because the Hamiltonian “time”-slicing might not preserve local bulk symmetries after gauge fixing. An example is provided by the holographic chiral anomalies discussed in [5], and in [27] in the Hamilton–Jacobi framework.

After the work of [16] there has been a number of papers discussing different aspects of the HRG using the Hamilton–Jacobi formalism for gravity. Most of them, however, elaborate gravity coupled to scalar fields [28, 29, 30, 31, 32, 19, 33, 34, 35], where the situation is by now well understood. In this article we shall extend this analysis to essentially more general situations. We consider a general setup in which spin- $\frac{1}{2}$ fermions and form fields interact with gravity. In particular the treatment of spinors is rather subtle. These fields are linear in momenta, couple derivatively to the metric and are in general endowed with a complex structure. It was already noticed in previous literature [36, 37, 17, 38] on the AdS/CFT correspondence that fermion fields split naturally in momentum and coordinate parts, and that the latter part are the relevant sources for the boundary operators. The split boils down, in general, to imposing some Lorentz-invariant condition on the fermions, such as chirality or reality conditions. In the following we systematize these observations in the context of Holography, using the Hamilton–Jacobi approach. The Ward identities resulting in this case from the gravitational constraints generalize the structures that arise from the scalar and Yang–Mills sectors, and display some intriguing novel features, as well.

The plan of the paper is as follows. In Section 2 we use the pure gravity system to illustrate how to implement and solve the descent equations that follow from the \mathcal{H}_\perp constraint. In the process we will present a novel way

⁷In that context form fields were considered in [26].

of calculating the holographic Weyl anomaly of [39]⁸. We make then general comments on how Ward identities and holographic anomalies arise. In Section 3 we briefly review the structure of the holographic Callan–Symanzik equation and notice that there exists a distinguished coordinate system in which the flow equations take the familiar form involving beta functions. Section 4 is devoted to explaining the Hamiltonian treatment of spin- $\frac{1}{2}$ fermions and form fields. Sections 5, 6, and 7 contain our main results. In Section 5 we present the bracket structure induced by the \mathcal{H}_\perp constraint, while in Section 6 we discuss the other Ward identities. In Section 7 we solve, under certain (mild) assumptions, the bracket equations. We find agreement with results concerning the pure gravity sector and some interesting constraints in the form field and fermion sectors. The last section contains conclusions and future perspectives. In the appendices we have relegated some useful formulae and notation, and an extensive analysis of the fermionic phase space.

2 Renormalization and Anomalies

In order to illustrate the procedure, we consider in this section pure gravity with a cosmological constant. We start by showing how to get a solution for local terms of the generating functional F . As a by-product we shall get an alternative derivation of the holographic Weyl anomaly computed in [39]. We then set the ground to discuss Ward identities and anomalies.

Equation (6) in this case reads

$$\frac{-\eta\kappa^2}{\sqrt{\hat{g}}} \left(g_{ik}g_{jl} - \frac{1}{d-1}g_{ij}g_{kl} \right) \frac{\delta F}{\delta g_{ij}} \frac{\delta F}{\delta g_{kl}} = \sqrt{\hat{g}} \left(\frac{1}{\kappa^2}R - \Lambda \right) \quad (12)$$

and, following the notation introduced in [40], we rewrite it as⁹

$$(F, F) = \mathcal{L}_0 + \mathcal{L}_2 . \quad (13)$$

We then expand S_{div} according to the degree of divergence of each local term. For the case of pure gravity this is equivalent to a derivative expansion, as is easily seen by a change of coordinates $x_i \rightarrow t^{-1}x_i$.

The local terms that diverge at the UV fixed point are those that contain up to $d-1$ derivatives; logarithmically divergent terms may arise in the non-local part of W . However, the most divergent ones come always multiplying

⁸A similar approach has recently appeared in [32].

⁹The subscript counts the number of derivatives.

a scale invariant term. This is typical of the structure that arises in effective actions that produce a conformal anomaly [41].

Let us set $F = S_{\text{div}} + W$ in (12) and expand order by order in t . In global scalings the metric behaves as $g_{ij} \rightarrow t^{-2}g_{ij}$. Given that

$$(\cdot, \cdot) \rightarrow t^d(\cdot, \cdot) \quad (14)$$

$$S_n \rightarrow t^{-d+n}S_n \quad (15)$$

$$W \rightarrow W - 2 \log t \, S_d , \quad (16)$$

and $(S_0, S_d) = 0$, the following descent equations have to be satisfied

$$\begin{aligned} \mathcal{L}_0 &= (S_0, S_0) \\ \mathcal{L}_2 &= 2(S_0, S_2) \\ 0 &= 2(S_0, S_4) + (S_2, S_2) \\ &\vdots \\ -2(S_0, W) &= 2(S_2, S_{d-2}) + \cdots + 2(S_{\frac{d}{2}-2}, S_{\frac{d}{2}+2}) + (S_{\frac{d}{2}}, S_{\frac{d}{2}}) . \end{aligned} \quad (17)$$

The zeroth order equation fixes the relationship between the cosmological constants in the bulk and on the boundary, see Eq. (71). The n 'th order equation is an equation for S_n in terms of S_2, \dots, S_{n-2} . In particular, the last equation¹⁰ then gives the trace of the bare stress-energy tensor. This is because the lowest order term S_0 acts through the bracket (\cdot, \cdot) essentially as a scale transformation. The trace anomaly at the conformal points, where the beta functions vanish and couplings go to their fixed point values J^* , is given by

$$\langle T \rangle = \frac{2\eta(d-1)}{\kappa^2 \hat{\Lambda}} \lim_{J \rightarrow J^*} \frac{1}{\sqrt{\hat{g}}} (S_0, W) . \quad (18)$$

In the UV limit this expression remains finite and reproduces the holographic Weyl anomaly of [39]. The results for $d = 2$ and $d = 4$ are¹¹

$$\langle T \rangle = \frac{1}{\kappa^4 \hat{\Lambda}} R \quad (19)$$

$$\langle T \rangle = \frac{k_1^2}{\hat{\Lambda}} (3R^{ij}R_{ij} - R^2) . \quad (20)$$

¹⁰In even boundary dimension. In odd dimensions it fixes also higher order terms, and nonlocal logarithmic terms do not arise, as is well known.

¹¹With $\eta = 1$, see equations (71) and (74).

The only Ward identity that one has to check in this case is diffeomorphism invariance, Eq. (7). It is trivially satisfied for the finite (bare) stress-energy tensor, because the counter terms are generally covariant in the boundary metric

$$\nabla_i \frac{\delta W}{\delta g_{ij}} = -\nabla_i T_{\text{div}}^{ij} = 0 . \quad (21)$$

If there are matter fields, apart from the additional Ward identities related to other local symmetries, even the diffeomorphism identities will be modified. In fact, in the presence of nonzero vacuum expectation values the CFT stress-energy tensor is not conserved, but obeys rather, for one scalar source, say,

$$\nabla^j \langle T_{ij} \rangle = -\langle \mathcal{O} \rangle \nabla_i J . \quad (22)$$

This should not be considered, however, as a diffeomorphism anomaly, but rather a feature of the spontaneous symmetry breaking induced by the nonzero vacuum expectation values of \mathcal{O} . This is in accord with the non-conservation of the Brown–York [42] quasi-local stress-energy tensor, as was previously noticed in [34]. The signature of a gravitational anomaly is, instead, stress-energy nonconservation with no vev’s turned on, analogously to what happens if there is a holographic chiral anomaly [5, 27]. For obtaining an explicit anomaly in pure gravity it would therefore be necessary to consider the appropriate Chern–Simons terms. Such a term would be, for instance, in the $(2, 0)$ theory in 6 dimensions the reduction on S^4 of the M-theory CS term

$$S_{\text{CS}} = \int d^7x \text{Tr}(\omega \wedge R \wedge R \wedge R) . \quad (23)$$

We shall see in Section 6 that in the presence of fermion fields this pattern will be somewhat modified.

3 The Holographic RG Equations

In this section we review how the holographic RG equations arise. Here, we do not restrict to pure gravity theories, but assume that the brackets in (13) have been modified to accommodate variations with respect to matter fields. Equation (13) reads

$$(S_{\text{div}}, S_{\text{div}}) + 2(S_{\text{div}}, W) + (W, W) = \mathcal{L} . \quad (24)$$

The second term on the LHS is a linear operator acting on W . (W, W) is a quadratic correction to the linearized RG, and the left-over piece gives additional non-homogeneous terms, whose leading contribution was shown in the previous section to give the Weyl anomaly. In fact, because of cancellations achieved in the descent equations (17), (24) can be rewritten

$$2(S_{\text{div}}, W) + (W, W) = \mathcal{O}(1) . \quad (25)$$

In order to get a Callan–Symanzik type equation, one can take the following steps [16]: Calculate n variations of (25) w.r.t. the sources and drop the contact terms containing more than one delta function, which would not contribute for operators evaluated at different points. Then letting the sources go to constant (in x_i) values, the metric assuming the form,

$$g_{ij} = \mu(t) \hat{\eta}_{ij} , \quad (26)$$

and eventually integrating once over dx^d one gets expressions of the form

$$\left(\mathcal{U} \mu \frac{\partial}{\partial \mu} + \beta_I \frac{\partial}{\partial J_I} \right) \langle \mathcal{O}_{I_1} \mathcal{O}_{I_2} \rangle + \sum_{n=1}^2 \partial_{I_i} \beta^{J_i} \langle \mathcal{O}_I \mathcal{O}_{J_i} \rangle = \mathcal{O}(t^d \log^2 t) , \quad (27)$$

where¹²

$$\beta_I = \left(\frac{1}{\sqrt{\hat{g}}} G_{IJ} \frac{\delta S_{\text{div}}}{\delta J_J} \right)_{J=J(t), g=\hat{\eta}} \quad (28)$$

$$\mathcal{U} \simeq \left(\frac{1}{\sqrt{\hat{g}}} g_{ij} \frac{\delta S_{\text{div}}}{\delta g_{ij}} \right)_{J=J(t), g=\hat{\eta}} . \quad (29)$$

Notice that these equations are taken at a finite cutoff t , *i.e.* for *bare* quantities. Would one allow the couplings to be space-time dependent the vanishing of the beta functions would correspond to the sources obeying d -dimensional Einstein equations that would follow from S_{div} . This could give rise in principle to boundary backgrounds that *are not* Minkowski, or its conformal completion.

Considering finally the Hamiltonian flow equations (10) it turns out that in the appropriate gauge the transverse coordinate plays the role of a parameter for the scale transformation induced by the metric. In general it is not

¹²We drop a numerical coefficient.

always true that a gravity solution has the interpretation of a RG flow: It depends in fact on the leading behaviour of the bulk fields near the boundary of AdS [20, 21]. In the Hamilton–Jacobi context this means that not every solution S_{div} of the constraints gives rise to physical flows [19]. However, for those that are physical, the flow equations read, after fixing the gauge $N_i = 0$

$$\dot{\mu} \simeq N \mathcal{U} \mu \quad (30)$$

$$\dot{J}_I \simeq N \beta_I . \quad (31)$$

The scale transformation depends parametrically on the cutoff t . Let us now use (30) to express (27) in terms of the variation of t , and choose the gauge $N = +\frac{1}{t}$. This yields

$$\begin{aligned} & \left(t \frac{\partial}{\partial t} + \beta_I \frac{\partial}{\partial J_I} \right) \langle \mathcal{O}_{I_1} \cdots \mathcal{O}_{I_n} \rangle \\ & + \sum_{i=1}^n \partial_{I_i} \beta^{J_i} \langle \mathcal{O}_{I_1} \cdots \mathcal{O}_{J_i} \cdots \mathcal{O}_{I_n} \rangle = \mathcal{O}(t^d \log^2 t) , \end{aligned} \quad (32)$$

and (31) can be written as

$$\beta_I = t \frac{d}{dt} J_I , \quad (33)$$

which is consistent with the beta functions defined before and with interpreting (32) as a bare RG equation. Furthermore the right hand side represents higher order terms that, in this spirit, are logarithmic corrections to scaling. This choice of gauge differs from the Fefferman–Graham [43] by a sign: The latter corresponds in fact to $N = -\frac{1}{t}$.

As is the case in QFT, the definition of the beta functions is not unique. They are fixed unambiguously near the fixed points, where their behaviour is universal, but their extrapolation at intermediate RG steps is not [35]. The definition given in [16] differs from that given above by the ratio of the rate of scale change (30). In fact, in that reference $N = 1$ and all quantities depend on μ . The fixed points are not affected, because the zeros of the beta function cannot be modified by dividing by \mathcal{U} . Also, it is well known that the bare RG equations and the renormalized ones have the same physical content.

4 Fermions and Form Fields in Hamiltonian Theory

Let us now turn to a generic $(d + 1)$ -dimensional theory with spinors, form fields and local diffeomorphism invariance. We will consider the most general two-derivative action (neglecting Chern–Simons terms) with quadratic fermion couplings consistent with gauge symmetry, namely $S = S_I + S_{II} + S_{III}$, where

$$S_I = \frac{1}{\kappa^2} \int d^{d+1}x \sqrt{g} \left(\tilde{R} - 2\eta \tilde{\nabla} \cdot \tilde{\nabla}_n n + 2\eta \tilde{\nabla} \cdot (n \operatorname{tr} K) - \kappa^2 \Lambda \right) \quad (34)$$

$$S_{II} = \int d^{d+1}x \sqrt{g} \left(\frac{1}{2\lambda^2} F_A F^A + F_A J^A \right) \quad (35)$$

$$S_{III} = \frac{1}{2} \int d^{d+1}x \sqrt{g} \left(\bar{\psi} M \not{D} \psi - (\not{D} \bar{\psi}) M \psi + 2\bar{\psi} Z_A \Gamma^A \psi \right) . \quad (36)$$

The fields appearing in these expressions are the following: The field strength $F = dA$ is an Abelian p -form and couples to the fermions through $J^A = \bar{\psi} \zeta \Gamma^A \psi$. The capital Latin letters refer to a multi-index of pertinent rank; summations include division by the factorial of the rank. There can be arbitrarily many fermion flavours, but we always suppress the index that would distinguish them. This action encaptures, and generalizes, many interesting features of the effective superstring actions. For instance, F could be thought of as a Ramond–Ramond field.

The nondynamical couplings ζ , Z_A and M mix fermion flavours, and are not assumed space-time constants, unless explicitly indicated. They satisfy suitable hermiticity conditions so that the action is always real. They can be thought of as the Yukawa couplings to higgsed scalar fields. As Z_A is not a dynamical field, we have assumed $Z_{0A} = 0$. The underlying space-time has a boundary to which we can associate a normal vector field n and the extrinsic curvature K . This vector field has constant norm $n \cdot n = \eta = \pm 1$, and is hence either temporal or spatial.

In order to go over to the Hamiltonian formalism, we need to choose a particular “time coordinate”, or “evolution parameter” whose tangent field we call e_0 . The bulk vielbeine e_μ^α decompose into boundary vielbeine L_i^α , the vector n , and the nondynamical degrees of freedom N and N^i . The connection on the boundary is obtained (see Appendix A) from that in the bulk by shifting the Christoffel symbols in such a way that the n becomes

a covariantly constant vector field on the boundary, and that the boundary basis is covariantly constant in the normal direction

$$\nabla_i n = 0 \quad (37)$$

$$n \cdot \nabla_i e_j = 0. \quad (38)$$

Given the time direction e_0 we can now reduce all tensor fields in terms of fields defined on the boundary. Isolating the kinetic terms in the action, *i.e.* terms involving derivatives along e_0 , the remainder is by definition the total Hamiltonian

$$S = \int dt (p \dot{q} - H) \quad (39)$$

where

$$p \dot{q} = \int d^d x \left(p^{i\alpha} \partial_t L_{i\alpha} + E^{\hat{A}} \partial_t A_{\hat{A}} + \bar{\chi} \partial_t \psi - \partial_t \bar{\psi} \chi \right) \quad (40)$$

$$H = \int d^d x \left(N \mathcal{H}_\perp + N^i \mathcal{H}_i + A_0^{\hat{A}} \mathcal{G}_{\hat{A}} + \varepsilon_{\alpha\beta} \mathcal{J}^{\alpha\beta} \right) \quad (41)$$

We have added here, by hand, the constraint that guarantees the freedom to choose the flat basis for vielbeine freely, *i.e.* the generator of local Lorentz transformations in the bulk, $\mathcal{J}^{\alpha\beta}$. For notation, that is standard, see Appendix A.

The physical phase space can now be easily read off from the above reformulation. It consists of the canonical pairs $(p_{\alpha}^i, L_j^{\beta})$, $(E^{\hat{A}}, A_{\hat{B}})$, $(\bar{\chi}^a, \psi_b)$, and $(\bar{\psi}^a, \chi_b)$. The fields $A_0^{\hat{A}}$, N and N^i are Lagrange multipliers that correspond to the first class constraints $\mathcal{G}_{\hat{A}}$, \mathcal{H}_\perp and \mathcal{H}_i , respectively. Due to fermions, that have a first order action principle, there are also second class constraints

$$\chi = \frac{1}{2} \eta \sqrt{\hat{g}} \Gamma^n M \psi. \quad (42)$$

In order to solve these constraints, we have to split the fermion phase space in some Lorentz invariant way into two parts, one of which we treat as the configuration space, and hence boundary fields, and the other of which are then the momenta, to be solved as functionals of the boundary fields in HJ theory. This is done in Appendix C by imposing a chirality condition; here we shall only point out some salient features.

Most of the calculations must be done case by case, but the emerging structures are similar. Having used the condition of choice to put the kinetic term in such a form (as in formulae (131) and (137)) that the coordinates and the momenta can be read off, there are always some left-over pieces:

- 1) There turns out to be a total time-derivative term $\partial_t G$. This term should be simply subtracted from the action, as argued in [17, 38]. Generally speaking the reason for this procedure is that the generating function arises also on the bulk side, eventually, from a path integral, which is going to be defined more fundamentally in Hamiltonian language. It has also been pointed out [36] that, as first order actions vanish on classical solutions, we would otherwise get a trivial result as far as the fermions are concerned.
- 2) There will also arise terms that involve a time derivative of the metric. This will naturally change the gravitational momentum, but in a way that is easily kept track of. After some algebra it turns out to be sufficient to simply shift by

$$\pi^{ij} \longrightarrow \pi^{ij} - \frac{1}{2} \dot{g}^{ij} G . \quad (43)$$

5 The Holographic Callan–Symanzik Equation

Having split the fermion phase space we are now in the position to write down the Hamilton–Jacobi equations for the full system. We will perform this analysis for Weyl fermions and assume that M is a space-time constant matrix.

It turns out that, provided there are no marginal operators Z present in the bulk and that the rank of the tensor field p is *odd*, the Hamilton–Jacobi equation originating from \mathcal{H}_\perp does indeed take the generally expected form

$$(F, F) = \mathcal{L} . \quad (44)$$

Where now

$$(F, F) = (F, F)_g + (F, F)_A + (F, F)_\varphi , \quad (45)$$

and the RHS of (44) is

$$\begin{aligned} \mathcal{L} = \sqrt{\hat{g}} \left(\frac{1}{\kappa^2} R - \Lambda + \frac{1}{2\lambda^2} F_{\hat{A}} F^{\hat{A}} + F_{\hat{A}} \bar{\varphi} \zeta \Gamma^{\hat{A}} \varphi \right. \\ \left. + \frac{1}{2} \bar{\varphi} M \hat{\mathcal{D}} \varphi - \frac{1}{2} (\hat{\mathcal{D}} \bar{\varphi}) M \varphi + \bar{\varphi} Z_{\hat{A}} \Gamma^{\hat{A}} \varphi \right). \end{aligned} \quad (46)$$

It is useful to define the following operators¹³

$$\mathcal{D} = \frac{1}{2} \bar{\varphi} \frac{\delta}{\delta \bar{\varphi}} - \frac{1}{2} \frac{\delta}{\delta \varphi} \varphi \quad (47)$$

$$\mathcal{D}^{ij} = \frac{\delta}{\delta g_{ij}} - \frac{1}{2} \hat{g}^{ij} \mathcal{D} \quad (48)$$

$$\mathcal{D}^{\hat{A}} = \frac{\delta}{\delta A_{\hat{A}}} + \bar{\varphi} \zeta M^{-1} \Gamma^{\hat{A}} \frac{\delta}{\delta \bar{\varphi}} + \frac{\delta}{\delta \varphi} M^{-1} \zeta \Gamma^{\hat{A}} \varphi \quad (49)$$

Had we also considered p even, the last equation would have been different: Then the operator $\mathcal{D}^{\hat{A}}$ would have contained terms with either no or two derivatives w.r.t. the fermion fields. Now the brackets can be written easily¹⁴

$$(F, H)_g = \frac{-\eta \kappa^2}{\sqrt{\hat{g}}} (g_{il} g_{jk} - \frac{1}{d-1} g_{ij} g_{kl}) (\mathcal{D}^{ij} F) (\mathcal{D}^{kl} H) \quad (50)$$

$$(F, H)_A = \frac{\eta \lambda^2}{2\sqrt{\hat{g}}} (\mathcal{D}^{\hat{A}} F) (\mathcal{D}_{\hat{A}} H) \quad (51)$$

$$(F, H)_{\varphi} = \frac{-\eta}{2\sqrt{\hat{g}}} \left(\frac{\delta F}{\delta \varphi} M^{-1} \hat{\mathcal{D}} \frac{\delta H}{\delta \bar{\varphi}} - (\hat{\mathcal{D}} \frac{\delta F}{\delta \varphi}) M^{-1} \frac{\delta H}{\delta \bar{\varphi}} \right). \quad (52)$$

The reason for the fact that also the fermionic momenta give rise to a bracket is easily seen in the case for Weyl fermions: The chiralities of the coordinates and the momenta are such that if we insert any operator even in Clifford matrices between them, $\bar{\pi} \mathcal{O}_{\text{even}} \varphi$, the result is nontrivial. Similarly, the non-trivial results for odd operators arise from insertions between either two coordinates, $\bar{\varphi} \mathcal{O}_{\text{odd}} \varphi$, or two momenta, $\bar{\pi} \mathcal{O}_{\text{odd}} \pi$.

This structure changes slightly if the bulk mass terms, or couplings to external form fields, Z , are included. If Z is even, such as a mass term, there will be an additional term

$$\delta_Z F + (F, F) = \mathcal{L} \quad (53)$$

¹³Derivatives act from the left, but not on fields included in the same operator.

¹⁴Here we have set Z to zero. See below.

where

$$\delta_Z = \frac{1}{\sqrt{\eta}} \left(\bar{\varphi} Z M^{-1} \frac{\delta}{\delta \bar{\varphi}} - \frac{\delta}{\delta \varphi} M^{-1} Z \varphi \right). \quad (54)$$

This addition preserves the flow equation form of the final Callan–Symanzik equations, however. Also, if Z is odd or M is not constant, there will be an additional quadratic piece in the fermion momenta, and the basic form of the brackets will be, eventually, unchanged.

If we are forced to give the initial data in terms of Weyl fermions, as we are assuming in this article, the requirement that the bulk action produce a QFT generating functional that obeys the Callan–Symanzik equation implies that bulk theories where fermions are coupled to dynamical *even* rank form fields do not possess a simple holographic dual.

Including higher order interactions could be a problem, because they would introduce higher powers of momenta, which would spoil the basic form of the brackets. However, we have seen above an encouraging rearrangement of terms, where the structure of the theory solves a similar problem. For instance, the form field kinetic term absorbs some four fermion couplings in the expression $(\mathcal{D}^{\hat{A}} F)^2$. We can indeed view the fermionic additions in $\mathcal{D}^{\hat{A}}$ as a covariantization of the flat derivative with respect to the form field \hat{A} . There is, therefore, reason to expect that at least in theories that are known to have a holographic dual but which contain four fermion interactions the additional symmetries, such as local supersymmetry or the $SL(2, \mathbb{Z})$ invariance in type IIB SUGRA, might arrange the fermion structure in such a way that the eventual higher fermion derivatives would still be manageable.

6 First Class Constraints as Ward Identities

In addition to the constraint \mathcal{H}_\perp treated above, we have to solve also the rest of the first class constraints $\mathcal{G}_{\hat{A}}$, \mathcal{H}_i and $\mathcal{J}^{\alpha\beta}$. Generally speaking, they will impose gauge symmetry, diffeomorphism invariance and local Lorentz symmetry on the boundary. Why this is not straight forwardly so is because, in the bulk, these constraints generate symmetry transformations through Dirac brackets. Due to the ansatz (9) in Hamilton–Jacobi formalism, they will act differently on F , thus resulting in conditions that do not impose necessarily the full off-shell symmetries of the bulk theory. As far as the Lorentz and the

gauge symmetries are concerned the geometrically expected actions are obtained. In case the of diffeomorphism invariance some additional constraints arise.

For instance, the fact that $A_{\hat{A}}$ enters the generating functional F only through its field strength is sufficient to guarantee $\mathcal{G}_{\hat{A}} = 0$. This simply reflects the fact that the boundary theory must have the same gauge symmetry as the bulk theory. A similar situation prevails as far as $\mathcal{J}^{\alpha\beta}$ is concerned: $n_{\underline{\alpha}}\mathcal{J}^{\alpha\beta} = 0$ just because $n_{\underline{\alpha}}L_i{}^{\underline{\alpha}} = 0$, and the rest of the components generate the expected action on vielbeine and spinors, through

$$\Lambda^{jk} = L^{[j}{}_{\underline{\alpha}} \frac{\delta}{\delta L_{k]\underline{\alpha}}} - \frac{1}{4} \left(\frac{\delta}{\delta \varphi} \Gamma^{jk} \varphi + \bar{\varphi} \Gamma^{jk} \frac{\delta}{\delta \bar{\varphi}} \right). \quad (55)$$

This constraint guarantees, therefore, local Lorentz invariance on the boundary. Choosing gauge and Lorentz invariant S_{div} will be enough to avoid anomalies in the boundary theory.

The situation is somewhat more involved when the constraints that guarantee diffeomorphism invariance are considered. This would mean that the effective action be invariant under translations generated by a vector field χ . Or, in other words, that shifting the fields \hat{A} and φ infinitesimally by their Lie-derivatives

$$\mathcal{L}_{\chi} \hat{A} = \iota_{\chi} \hat{F} + \text{d} \iota_{\chi} \hat{A} \quad (56)$$

$$\mathcal{L}_{\chi} \varphi = (\hat{D}_{\chi} + \frac{1}{4} \nabla_{[i} \chi_{j]} \Gamma^{ij}) \varphi \quad (57)$$

we get a contribution that combines together with a contribution from the integration measure to a total derivative of the Lagrangian. Note that a general spinorial Lie-derivative does not obey the Leibniz rule¹⁵. It is therefore useful to restrict to Killing fields $\nabla_{(i} \chi_{j)} = 0$, for which the formula (57) applies.

Solving the constraint $\mathcal{H}_i = 0$ we get, again, assuming p odd and the Weyl decomposition, that the variation

$$\delta_{\chi} = \nabla_k \chi^j L_{j\underline{\alpha}} \frac{\delta}{\delta L_{k\underline{\alpha}}} + \chi^i \left(\partial_{[i} A_{\hat{A}]} \mathcal{D}^{\hat{A}} - \frac{\delta}{\delta \varphi} \hat{D}_i \varphi + \hat{D}_i \bar{\varphi} \frac{\delta}{\delta \bar{\varphi}} \right) \quad (58)$$

should annihilate the effective action. This differs from a Lie-derivative in two respects: First, the transformation of the form field is accompanied by

¹⁵For a general introduction to spinors and geometry see for instance [44].

a gauge transformation. Second, its action on fermions is modified by

$$\Delta_\chi \varphi = -\iota_\chi F_{\hat{A}} \Gamma^{\hat{A}} M^{-1} \zeta \varphi \quad (59)$$

$$\Delta_\chi \bar{\varphi} = \iota_\chi F_{\hat{A}} \bar{\varphi} \Gamma^{\hat{A}} \zeta M^{-1} \quad (60)$$

where $\Delta_\chi = \delta_\chi - \mathcal{L}_\chi$. If we want to restrict to theories where the boundary diffeomorphism invariance still prevails, we have to put this difference to zero. A solution of $\Delta_\chi S_{\text{div}} = 0$ is ensured imposing the following conditions:

- 1) The Clifford action of any differential form K appearing in the fermion couplings $\bar{\varphi} K_{\hat{A}} \Gamma^{\hat{A}} \varphi$ commute with that of $\iota_\chi \hat{F}$, *i.e.*

$$[K_{\hat{A}} \Gamma^{\hat{A}}, \chi^i F_{i\hat{B}} \Gamma^{\hat{B}}] = 0 . \quad (61)$$

For instance, for $K_{\hat{A}} = F_{\hat{A}}$, $F_{\hat{A}}$ being of odd rank, this is clearly true. This condition means then, geometrically, that $K_{\hat{A}}$ and $F_{\hat{A}}$ should be aligned in a certain way. In addition to this the couplings should satisfy, in the notation of formula (70),

$$\hat{\zeta} M^{-1} \zeta = \zeta M^{-1} \hat{\zeta} . \quad (62)$$

- 2) If there is a kinetic term on the boundary, such as $\bar{\varphi} \hat{\mathcal{D}} \varphi - (\hat{\mathcal{D}} \bar{\varphi}) \varphi$ the following restrictions be true

$$\mathcal{L}_\chi \hat{F} = 0 \quad (63)$$

$$\iota_\chi F^{i\hat{A}} \hat{D}_i \varphi = 0 \quad (64)$$

These are strong requirements, as they concern the boundary fields and not only couplings, and therefore really restrict, from the bulk point of view, the set of acceptable initial conditions. The simplest way to solve them is naturally to exclude the kinetic terms from the action, cf. end of Section 7. However, this might be too drastic a solution, as, in the above, we are assuming that there exist a Killing field χ ; after all, we are considering an interacting theory with physical sources. More interestingly, these conditions can be solved by assuming $\iota_\chi \hat{F} = 0$: This means that the Killing isometry only changes the field \hat{A} by generating a gauge transformation. With this understanding it is sufficient to set

$$\mathcal{L}_\chi \hat{A} \approx 0 . \quad (65)$$

This would mean that there are no restrictions on the fermion fields, whereas the form field potential is frozen to configurations covariant under flows generated by χ .

Then, considering diffeomorphism invariant terms in S_{div} , one obtains the Ward identity

$$\mathcal{L}_\chi W + \Delta_\chi \bar{\varphi} \langle \mathcal{O}_{\bar{\varphi}} \rangle - \langle \mathcal{O}_\varphi \rangle \Delta_\chi \varphi = -\Delta_\chi S_{\text{div}} . \quad (66)$$

The RHS is the failure of the counter terms to satisfy the constraint $\mathcal{H}_i = 0$, while the vacuum expectation values would signal the spontaneous symmetry breaking of Lorentz symmetry in the dual QFT. This is analogous to the analysis in Sec. 2 where scalar fields coupled to gravity were considered.

7 Local Expansion in the Bracket Equation

Let us consider a particular solution of the classical equations of motion that behaves at the boundary $t \rightarrow 0$ as

$$g_{ij}(x, t) = t^{-2} g_{ij}^R(x) + \mathcal{O}(t^{-1}) \quad (67)$$

$$A_{\hat{A}}(x, t) = t^\nu A_{\hat{A}}^R(x) + \mathcal{O}(t^{\nu+1}) \quad (68)$$

$$\psi(x, t) = t^\sigma \psi^R(x) + \mathcal{O}(t^{\sigma+1}) \quad (69)$$

in the coordinate system of Eq. (11). The quantities with superscript R refer to expressions that have a finite and nonzero limit at $t \rightarrow 0$. We will actually not need to show whether the full coupled system has solutions with this particular asymptotic behaviour. This would also be quite difficult — it was found, for instance, in [17, 38] that free fermions scale as $\sigma = d/2 - m$, where m is the bulk mass. In our case the coupling of the form field F_A to the fermions gives rise to an effective mass term. We cannot, however, fix the field F_A in any useful way in order to analyse conclusively the scaling behaviour in this coupled system.

Instead, we impose the above scaling behaviour for some set of critical exponents ν and σ and then derive from this – and the assumption that the holographic dual exist at all – consistency conditions for both the bulk and the boundary theories. Let us consider, in particular, the case $\nu = -n + 1$ and $\sigma = 1/2$. This assignment of critical exponents has the virtue that the expansion of S_{div} will look like an expansion in terms of the naive mass

dimension relevant to supersymmetric models. This choice will turn out to be a convenient book-keeping device, but the results we eventually get are valid more generally. In order to solve the full bracket equation, we will have to arrange the coefficients of terms that not only have the same scaling behaviour, but also the same structure, to cancel. So, in principle, one can check *a posteriori* for which range of scaling exponents the terms we neglected are still subleading and our results continue to be then valid. We shall further assume that all couplings such as M and Z are marginal operators, and as such t -independent. This assumption is not very restrictive, not even in the presence of scalar fields, that would render the couplings dynamical. We can now write a local ansatz for S_{div} that is of the same functional form as (46)

$$S_{\text{div}} = \int d^d x \sqrt{\hat{g}} \left(k_1 R - \hat{\Lambda} + \frac{1}{2} k_2 F_{\hat{A}} F^{\hat{A}} + F_{\hat{A}} \bar{\varphi} \hat{\zeta} \Gamma^{\hat{A}} \varphi + \frac{1}{2} \bar{\varphi} \hat{M} \hat{\mathcal{D}} \varphi - \frac{1}{2} (\hat{\mathcal{D}} \bar{\varphi}) \hat{M} \varphi + \bar{\varphi} L_{\hat{A}} \Gamma^{\hat{A}} \varphi \right) . \quad (70)$$

As we shall later restrict to a scalar coupling L , it turns out that no four fermion terms are needed. The couplings $k_1, k_2, \hat{\zeta}, \hat{M}$ and $L_{\hat{A}} \Gamma^{\hat{A}}$ need not be constant on the boundary, but they are assumed marginal, *i.e.* time-independent.

At leading order S_{div} will diverge as t^{-d} when $t \rightarrow 0$. We shall consider the three lowest order contributions to the equation $\mathcal{H}_{\perp} = 0$, or, $(S_{\text{div}}, S_{\text{div}}) = \mathcal{L}$.

- 1) At the leading order we get

$$\Lambda + \frac{\eta \kappa^2}{4} \frac{d}{d-1} \hat{\Lambda}^2 = 0 . \quad (71)$$

The boundary cosmological constant is therefore essentially the square root of the bulk one and sets the scale. We notice that depending on the sign of the bulk gravitational constant we can choose to look at either time-like or space-like boundary surfaces. However, only the space-like surfaces admit an asymptotically anti de Sitter solution.

- 2) The $\mathcal{O}(t^{-d+1})$ equations depend on the details of Z . The equations simplify assuming that Z is odd in Clifford matrices and that L is a scalar, *i.e.* a flavour matrix, as we then get

$$0 = Z + \eta L M^{-1} Z M^{-1} L . \quad (72)$$

If Z is even, we get

$$ZM^{-1}L + LM^{-1}Z = 0. \quad (73)$$

In particular, we could consistently set $L = 0$. This would mean that, for instance, a bulk mass term does not automatically lead to a mass term on the boundary.

- 3) At the next order $\mathcal{O}(t^{-d+2})$ we get from the Einstein–Hilbert and the form field kinetic terms conditions for the couplings k_1 and k_2

$$\frac{\eta}{d-1}\kappa^2\hat{\Lambda}\left(\frac{d}{2}-1\right)k_1 - \frac{1}{\kappa^2} = 0 \quad (74)$$

$$\frac{\eta}{d-1}\kappa^2\hat{\Lambda}\left(\frac{d}{2}-p\right)k_2 - \frac{1}{\lambda^2} = 0. \quad (75)$$

For $d > 2$ equation (74) allows one to compute the coefficient of the R term, as expected. In $d = 2$ this equation does not arise as $\sqrt{\hat{g}}R$ is marginal, and it is not to be included in S_{div} . Instead, the impossibility of canceling it in the descent equations translates to the Weyl anomaly in $d = 2$ as in Eq. (19).

Equation (75) gives the value of k_2 for $d \neq 2p$. Note, however, that for middle-dimensional form fields $d = 2p$ this equation will not have solutions. In the case of free form fields, as they are always marginal [26], it is just the analogue of (74) for pure gravity: In this case $F^{\hat{A}}F_{\hat{A}}$ is a marginal operator and it contributes to the matter part of the Weyl anomaly. In the interacting case the treatment of this term depends on the scaling dimension ν . If $\nu < 0$ then the contribution should really be included in the divergent part, and a middle-dimensional form field would not allow consistent solutions for the bracket equation.

- 4) Assuming that M and L are constants on the boundary, the fermion terms yield the constraints

$$-\frac{p}{2}\frac{\eta}{d-1}\kappa^2\hat{\Lambda}\hat{M} = M + \eta LM^{-1}L \quad (76)$$

$$-\frac{\eta}{2d-2}\kappa^2\hat{\Lambda}\hat{\zeta} = \zeta + \eta LM^{-1}\zeta M^{-1}L. \quad (77)$$

In all of the equations (71) and (74) – (77) we see that the relationship between the bulk and the boundary couplings is essentially a scale factor $\kappa^2\hat{\Lambda}$.

It is interesting to note that the role of L is to mix bulk fermion flavours into new combinations on the boundary. Assuming L proportional to M would lead to scaling the fermion field. Furthermore, setting $L = \sqrt{-\eta}M$ the dynamical part to the fermion action drops out completely $\hat{M} = \hat{\zeta} = 0$, and the only contribution comes from the L -term itself

$$\sqrt{-\eta} \int d^d x \sqrt{\hat{g}} \bar{\varphi} M \varphi . \quad (78)$$

As discussed in the previous section this solution would not produce a diffeomorphism anomaly, even without restricting to symmetric form fields, cf. Eq. (65).

8 Conclusions

We have investigated the relationship between diffeomorphism invariant theories and their holographic duals, showing in particular that, in a theory that contains fermions, nontrivial consistency conditions arise. These conditions restrict, for instance, the couplings of even rank form field strengths to fermions.

After explaining how the Hamiltonian reduction is performed, with particular attention to fermion fields, in Sections 5 and 6 we have discussed how the holographic Callan–Symanzik equation and other Ward identities following from first class constraints get modified in the presence of fermions and forms. Although the gauge and the Lorentz constraints did not lead to surprises, the Poincaré constraints resulted in an anomalous contribution in the diffeomorphism Ward identity on the boundary. However, one can get rid of these terms, either imposing conditions on the sources, as for instance the boundary gauge potential to be constant in the sense of Eq. (65), or by the choice of a suitable counter term (78).

We have also derived relationships between the bulk and boundary couplings, finding that the role of the cosmological constant on the boundary is to set the scale of all boundary couplings w.r.t bulk couplings, as expected. Moreover, we were able to find the first terms of the expansion of the counter term action S_{div} , with couplings fixed in terms of those appearing in the bulk action. This analysis shows also that one can avoid including fermion kinetic terms in S_{div} that would have a nonzero anomalous diffeomorphism variation.

There still remain open problems. One should clarify, for instance, what kind of restrictions higher fermion couplings impose on the duals, or what role other bulk symmetries, such as supersymmetry, might play in the bracket structure. In particular, it would be interesting to include dynamical scalar fields and Rarita–Schwinger fermions in these considerations. One would also like to know how robust our results actually are when the scaling behaviour of the bulk fields at the boundary are varied. Finally, the formal structures arising here are quite intriguing: It would be interesting to find out whether the (\cdot, \cdot) -bracket or the \mathcal{D} -operators have a geometrical meaning in solving Ward identities.

Acknowledgements: We thank C. Acatrinei, M. Henningson, R. Iengo, and A. Mukherjee for discussions. We are grateful to G. Ferretti for useful comments and encouragement. D.M. would like to thank the department of Physics of Chalmers University (Göteborg) for hospitality during the latest stages of this work. D.M. has received financial support from Chalmers University of Göteborg. This work was supported in part by the EC TMR program CT960045.

A Notation and useful formulae

Greek indices μ, ν, \dots refer to the coordinate directions in the bulk, underlined Greek indices $\underline{\alpha}, \underline{\beta}, \dots$ are flat indices in the bulk, lower case Latin indices i, j, \dots refer to coordinate directions in the boundary, upper case Latin indices A, B, \dots refer to normalized multi-indices in the bulk, and hatted upper case Latin indices \hat{A}, \hat{B}, \dots refer to normalized multi-indices in the boundary. If there is danger of confusion, symbols with a tilde, such as $\tilde{\nabla}$, are used to refer to bulk quantities and symbols with a hat, such as $\hat{\mathcal{D}}$, to boundary quantities.

We can express the connection between an arbitrary flat coordinate basis $\{e_{\underline{\alpha}}\}$ and a basis $\{e_0, e_i\}$ that involves the direction of the evolution coordinate (bulk direction) e_0 in terms of the vielbeine

$$e_i^{\underline{\alpha}} = L_i^{\underline{\alpha}} \tag{79}$$

$$e_0^{\underline{\alpha}} = N n^{\underline{\alpha}} + N^i L_i^{\underline{\alpha}}. \tag{80}$$

Due to the algebraic constraint $L_i{}^\alpha n_\alpha = 0$ we have $e_i \cdot n = 0$. For other properties of this frame see for instance [45]. The boundary metric and vielbeine are related through

$$g_{ij} = L_i{}^\alpha L_j{}^\beta \eta_{\alpha\beta} \quad (81)$$

$$\eta_{\alpha\beta} = L_{i\alpha} L^i{}_\beta + \eta n_\alpha n_\beta, \quad (82)$$

and the boundary gamma matrices are defined by

$$\Gamma^n = n_\alpha \Gamma^\alpha \quad (83)$$

$$\Gamma^i = L^i{}_\alpha \Gamma^\alpha. \quad (84)$$

Given the Levi-Civita connection $\tilde{\nabla}$ in the bulk we can construct a metric connection on the boundary by setting [46]

$$\nabla_X Y = \tilde{\nabla}_X Y + \eta n (Y \cdot \tilde{\nabla}_X n) - \eta \tilde{\nabla}_X n (Y \cdot n) \quad (85)$$

for arbitrary vector fields X, Y . This connection enjoys the properties

$$\nabla_i n = 0 \quad (86)$$

$$n \cdot \nabla_i e_j = 0. \quad (87)$$

The spin connection in the bulk can be expressed in terms of that on the boundary using

$$\begin{aligned} \tilde{\omega}_{i\alpha\beta} &= \Gamma_{jik} L_\alpha^j L_\beta^k - L_\beta^k \partial_i L_{k\alpha} - \eta n_\beta \partial_i n_\alpha \\ &\quad + \eta K_{ij} (n_\alpha L_\beta^j - n_\beta L_\alpha^j) \end{aligned} \quad (88)$$

$$\begin{aligned} \tilde{\omega}_{0\alpha\beta} &= (\partial_j N + \eta N^i K_{ij}) (n_\alpha L_\beta^j - n_\beta L_\alpha^j) \\ &\quad - \nabla_{[j} N_{k]} L_\alpha^j L_\beta^k + \eta n_{[\alpha} \partial_t n_{\beta]} + L_{j[\alpha} \partial_t L_{\beta]}^j. \end{aligned} \quad (89)$$

The extrinsic curvature is

$$K_{ij} = -\frac{1}{2N} (\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i). \quad (90)$$

From the point of view of the Lagrangian formalism, the momenta are just notation for expressions involving fields and their derivatives

$$p^{i\alpha} = 2\pi^{ij} L_j^\alpha - \frac{1}{8} \eta \sqrt{\hat{g}} \bar{\psi} M \{ \Gamma^n, \Gamma^{\alpha\beta} \} \psi L_\beta^i \quad (91)$$

$$\pi^{ij} = -\frac{\eta}{\kappa^2} \sqrt{\hat{g}} (\hat{g}^{ij} \text{tr} K - K^{ij}) \quad (92)$$

$$E^{\hat{A}} = \sqrt{g} \left(\frac{1}{\lambda^2} F^{0\hat{A}} + J^{0\hat{A}} \right) \quad (93)$$

$$\bar{\chi} = \frac{1}{2} \eta \sqrt{\hat{g}} \bar{\psi} \Gamma^n M \quad (94)$$

$$\chi = \frac{1}{2} \eta \sqrt{\hat{g}} \Gamma^n M \psi . \quad (95)$$

The Poincaré constraints consist of three parts $\mathcal{H}^\mu = \mathcal{H}_I^\mu + \mathcal{H}_{II}^\mu + \mathcal{H}_{III}^\mu$. In the pure gravity sector we have

$$\mathcal{H}_I^\perp = \sqrt{\hat{g}} \left(-\frac{1}{\kappa^2} R + \Lambda \right) - \frac{\eta \kappa^2}{\sqrt{\hat{g}}} \left(\text{tr} \Pi^2 - \frac{1}{d-1} (\text{tr} \Pi)^2 \right) \quad (96)$$

$$\mathcal{H}_I^i = -\nabla_j (P^{j\alpha} L_\alpha^i) , \quad (97)$$

where the gravitational momenta have been shifted according to

$$\Pi^{ij} = \pi^{ij} - \frac{1}{2} \hat{g}^{ij} G \quad (98)$$

$$P^{j\alpha} = p^{j\alpha} - L^{i\alpha} G . \quad (99)$$

In the form field sector

$$\begin{aligned} \mathcal{H}_{II}^\perp &= \sqrt{\hat{g}} \left(-\frac{1}{2\lambda^2} F_{\hat{A}} F^{\hat{A}} - F_{\hat{A}} J^{\hat{A}} \right) \\ &\quad + \frac{1}{\sqrt{\hat{g}}} \frac{\eta \lambda^2}{2} (E_{\hat{A}} - \sqrt{g} J_{\hat{A}}^0) (E^{\hat{A}} - \sqrt{g} J^{0\hat{A}}) \end{aligned} \quad (100)$$

$$\mathcal{H}_{II}^i = F^i_{\hat{A}} (E^{\hat{A}} - \sqrt{g} J^{0\hat{A}}) , \quad (101)$$

the different signs in front of the fermionic dynamical (101) and background (103) currents is not a surprise, as a contribution of the first one has been used in the definition of the electric field $E_{\hat{A}}$.

In the fermion sector

$$\mathcal{H}_{III}^\perp = \frac{1}{2} \sqrt{\hat{g}} \left(-\bar{\psi} M \hat{D} \psi + (\hat{D} \bar{\psi}) M \psi - 2 \bar{\psi} Z_{\hat{A}} \Gamma^{\hat{A}} \psi \right) \quad (102)$$

$$\mathcal{H}_{III}^i = \frac{1}{2} \eta \sqrt{\hat{g}} \left(\bar{\psi} \Gamma^n M \hat{D}^i \psi - (\hat{D}^i \bar{\psi}) \Gamma^n M \psi + 2 \bar{\psi} Z_{\hat{A}}^i \Gamma^n \Gamma^{\hat{A}} \psi \right) \quad (103)$$

The action of the covariant derivative on spinors is, by definition

$$\not{D}\psi = \Gamma^\mu(\partial_\mu + \frac{1}{4}\omega_{\underline{\mu}\underline{\alpha}\underline{\beta}}\Gamma^{\underline{\alpha}\underline{\beta}})\psi \quad (104)$$

$$(\not{D}\bar{\psi}) = (\partial_\mu\bar{\psi} - \frac{1}{4}\bar{\psi}\omega_{\underline{\mu}\underline{\alpha}\underline{\beta}}\Gamma^{\underline{\alpha}\underline{\beta}})\Gamma^\mu, \quad (105)$$

so that

$$\tilde{\nabla}_\mu(\bar{\chi}\Gamma^\mu\psi) = (\not{D}\bar{\chi})\psi + \bar{\chi}\not{D}\psi. \quad (106)$$

In addition to the Poincaré constraints there are also constraints that generate the gauge transformations $A \longrightarrow A + dB$ and gauge transformations on the frame bundle (local Lorentz transformations)

$$\mathcal{G}^{\hat{A}} = \partial_i E^{i\hat{A}} \quad (107)$$

$$\mathcal{J}^{\underline{\alpha}\underline{\beta}} = p^{i[\underline{\alpha}}L_i^{\underline{\beta}]} + \frac{1}{8}\eta\sqrt{\hat{g}}\bar{\psi}M\{\Gamma^n, \Gamma^{\underline{\alpha}\underline{\beta}}\}\psi. \quad (108)$$

B Clifford algebra

The formulae in this appendix are taken mostly from Ref. [47].

B.1 Even dimensions

Consider a metric with signature $\eta_{ab} = \{(-)^{d_-}, (+)^{d_+}\}$, such that $d = d_- + d_+ = 2m$. The Clifford algebra is span by

$$\{\Gamma^i, \Gamma^j\} = 2g^{ij}. \quad (109)$$

All representations are unitarily equivalent, and the intertwining operators are

$$\Gamma_i^\dagger = A\Gamma_i A^{-1} \quad (110)$$

$$-\Gamma_i^T = C^{-1}\Gamma_i C \quad (111)$$

$$-\Gamma_i^* = D^{-1}\Gamma_i D \quad (112)$$

$$\Gamma_i^* = \tilde{D}^{-1}\Gamma_i \tilde{D} \quad (113)$$

where $D = CA^T$ and $\tilde{D} = \Gamma CA^T$. The chirality operator is

$$\Gamma = \Gamma^1 \cdots \Gamma^d \quad (114)$$

We have the following phases

$$A = \alpha A^\dagger \quad (115)$$

$$C = \tilde{\eta} C^T \quad (116)$$

$$DD^* = \delta \quad (117)$$

$$\tilde{D}\tilde{D}^* = \tilde{\delta} \quad (118)$$

where $|\alpha| = 1$, $\tilde{\eta} = \pm 1$, and $\delta^* = \tilde{\delta}$. The Dirac conjugate is defined as $\bar{\psi} = \psi^\dagger A$, and the charge conjugate as $\psi^c = CA^T\psi^*$, or $\psi^c = \Gamma CA^T\psi^*$. The chirality matrix satisfies

$$\Gamma^\dagger = (-)^m A\Gamma A^{-1} \quad (119)$$

$$\Gamma^T = (-)^m C^{-1}\Gamma C. \quad (120)$$

B.2 Odd dimensions

In this appendix we build a representation of an odd-dimensional $D = d + 1$ Clifford algebra with $(\Gamma^n)^2 = \eta$ starting from a given even dimensional $d = 2m$ Clifford algebra. We actually only need to construct the correct Clifford matrix Γ^n

$$\Gamma^n = \sqrt{(-)^{m+d_-}\eta\hat{g}} \Gamma. \quad (121)$$

In odd dimensions not all representations are equivalent. Instead, we only have the intertwining operators

$$\Gamma_{\underline{\alpha}}^\dagger = (-)^{d_-}\eta \tilde{A}\Gamma_{\underline{\alpha}}\tilde{A}^{-1} \quad (122)$$

$$\Gamma_{\underline{\alpha}}^T = (-)^m \tilde{C}^{-1}\Gamma_{\underline{\alpha}}\tilde{C} \quad (123)$$

$$\Gamma_{\underline{\alpha}}^* = (-)^{m+d_-}\eta \mathcal{D}^{-1}\Gamma_{\underline{\alpha}}\mathcal{D} \quad (124)$$

where $\mathcal{D} = \tilde{C}\tilde{A}^T$, not to be confused with (47). There are two inequivalent conjugacy classes: which representations belong to which depends on m, d_- and η .

We can represent the odd-dimensional intertwinors in terms of their even dimensional counter parts as

$$(-)^{d_-}\eta = \begin{cases} +1, & \tilde{A} = A \\ -1, & \tilde{A} = A\Gamma^n \end{cases} \quad (125)$$

$$(-)^m = \begin{cases} +1, & \tilde{C} = \Gamma^n C \\ -1, & \tilde{C} = C \end{cases} \quad (126)$$

We sometimes abbreviate the sign $(-)^{d-}\eta$ by ε . The operators \mathcal{D} can be expressed in terms of boundary operators

\mathcal{D}	$(-)^m = 1$	$(-)^m = -1$
$(-)^{d-}\eta = 1$	\tilde{D}	D
$(-)^{d-}\eta = -1$	ηD	$-\tilde{D}$

(127)

where $D = CA^T$ and $\tilde{D} = \Gamma^n D$.

C Fermion phase space

Fermions can be decomposed in bulk and boundary components in essentially two Lorentz invariant ways, namely by using chirality or reality conditions. Here we consider only chirality conditions, with which we refer to the eigenvalues ± 1 of $\sqrt{\eta}\Gamma^n$. This is essentially the only Clifford matrix whose eigenvalues we can consider without breaking Lorentz invariance explicitly.

We have to divide our analysis in two cases depending on the sign η and details of the bulk metric. Defining $\sqrt{\eta}\Gamma^n \psi_{\pm} = \pm \psi_{\pm}$ we get

$$\bar{\psi}_{\pm} \sqrt{\eta}\Gamma^n = \pm \varepsilon \eta \bar{\psi}_{\pm}, \quad (128)$$

where ε is the sign that appears in the relation of the Clifford matrices to their Hermitian conjugates (122). This means that the Dirac dual of a spinor of definite chirality is either of the same or the opposite chirality; as a consequence, the Lagrangian fields that correspond to the phase space coordinates will be different. It is useful to note that

$$\varepsilon \eta = \begin{cases} (-)^{d-} & d+1 \text{ odd} \\ \eta & d+1 \text{ even} \end{cases} \quad (129)$$

Case I. Assume $\varepsilon \eta = 1$. The kinetic term separates into

$$-\sqrt{\eta}\hat{g}(\bar{\psi}_- M \dot{\psi}_- + \bar{\psi}_+ M \dot{\psi}_+) - \partial_t G \quad (130)$$

$$-\frac{1}{2}\bar{\psi}_- \partial_t(\sqrt{\eta}\hat{g}M) \psi_- - \frac{1}{2}\bar{\psi}_+ \partial_t(\sqrt{\eta}\hat{g}M) \psi_+ \quad (131)$$

where

$$G = -\frac{1}{2}\sqrt{\eta\hat{g}}(\bar{\psi}_-M\psi_- + \bar{\psi}_+M\psi_+) . \quad (132)$$

The fermionic phase space consists therefore of the symplectic pairs $(\varphi, \bar{\pi})$ and $(\bar{\varphi}, \pi)$, where $\varphi = \psi_-$ and $\bar{\varphi} = \bar{\psi}_+$ and

$$\bar{\pi} = -\sqrt{\eta\hat{g}} \bar{\psi}_-M \quad (133)$$

$$\pi = \sqrt{\eta\hat{g}} M\psi_+ . \quad (134)$$

The last two terms in (131) produce a term

$$\frac{1}{2}\hat{g}^{ij}G \partial_t g_{ij} \quad (135)$$

in the action, and therefore cause a shift in the gravitational momentum.

Case II. Assume $\varepsilon\eta = -1$. The kinetic term separates into

$$-\sqrt{\eta\hat{g}}(\dot{\bar{\psi}}_-M\psi_+ + \bar{\psi}_+M\dot{\psi}_-) - \partial_t G \quad (136)$$

$$-\frac{1}{2}\bar{\psi}_-\partial_t(\sqrt{\eta\hat{g}}M) \psi_+ - \frac{1}{2}\bar{\psi}_+\partial_t(\sqrt{\eta\hat{g}}M) \psi_- \quad (137)$$

where

$$G = -\frac{1}{2}\sqrt{\eta\hat{g}}(\bar{\psi}_-M\psi_+ + \bar{\psi}_+M\psi_-) . \quad (138)$$

The configuration space is span by $\varphi = \psi_-$ and $\bar{\varphi} = \bar{\psi}_-$, and the momenta are

$$\bar{\pi} = -\sqrt{\eta\hat{g}} \bar{\psi}_+M \quad (139)$$

$$\pi = \sqrt{\eta\hat{g}} M\psi_+ . \quad (140)$$

Notice that, due to (125), the Dirac conjugates $\bar{\varphi}$ and $\bar{\psi}_-$ are formed differently: the former using the matrix A and the latter with \tilde{A} . This will result in an extra Γ^n everywhere, including the kinetic term, and the resulting extra sign hence cancels out. The gravitational momenta are shifted as in Case I.

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